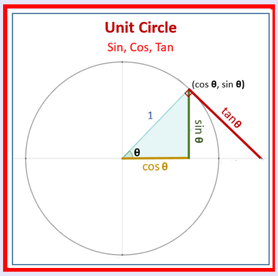
Math 241

Winter 2024

Lecture 15



Feb 19-8:47 AM

$f(x) \to f$ of $x \to$ function of x

$y = f(x)$

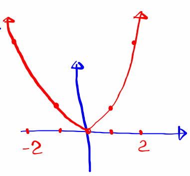
Range

Domain

For every x-value, there is only one y-value.

ex: $f(x) = x^2$

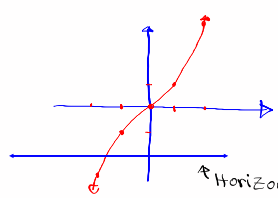
x	y
0	0
1	1
-1	1
2	4
-2	4



If the graph of $y = f(x)$ is an increasing or a decreasing graph (Not both), then $f(x)$ is called a one-to-one function.

$f(x) = x^3$

x	y
0	0
1	1
-1	-1
2	8
-2	-8



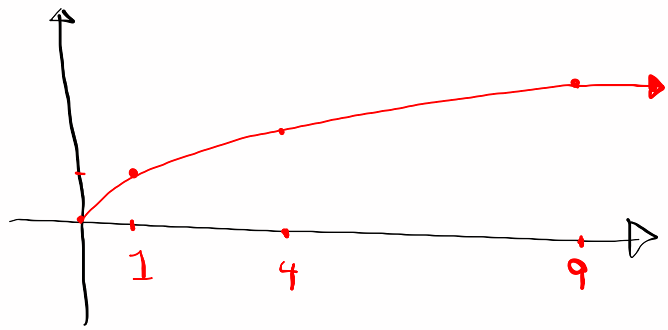
Horizontal line Test

In a one-to-one function, every y-value is only used once.

Jan 29-8:01 AM

$f(x) = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3



It is increasing only.

It passes the horizontal line Test.

It is one-to-one function.

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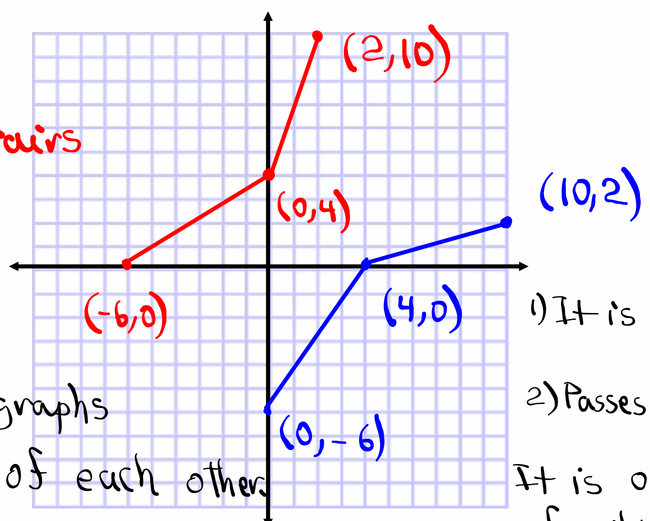
Consider the graph below:

If we switch all ordered-Pairs



These two graphs are inverse of each other.

1) It is increasing

2) Passes the HLT

It is one-to-one function.

Jan 29-8:13 AM

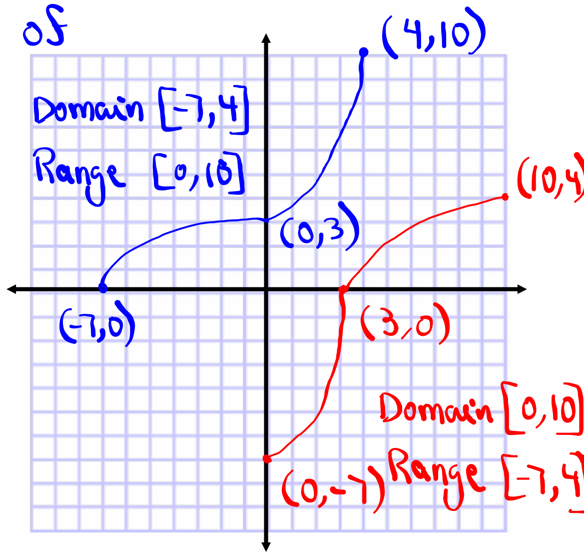
Graph the inverse of the display below.

x $\quad$ y

Domain → Range

Range → Domain

y $\quad$ x



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Inverse notation

Function $\quad f(x)$

Not exponent
Does not mean reciprocal

Inverse $\quad f^{-1}(x)$ $\quad$ "f - inverse of x"

$f(x) = \sin x$

$f^{-1}(x) = \sin^{-1} x$ $\quad$ "Sine-inverse of x"

$f(x) = \cos x$

$f^{-1}(x) = \cos^{-1} x$ $\quad$ "Cosine-inverse of x"

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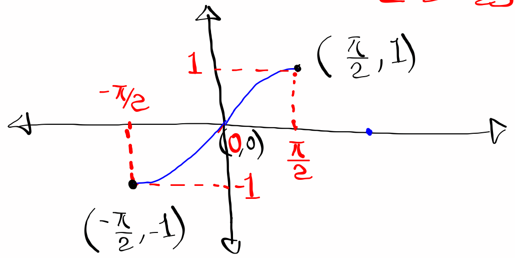
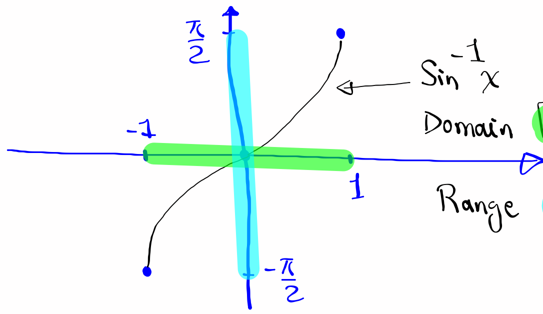
Let's graph

$f(x) = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Let's switch all ordered-pairs.



$\sin^{-1} x$

Domain $[-1, 1]$

Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

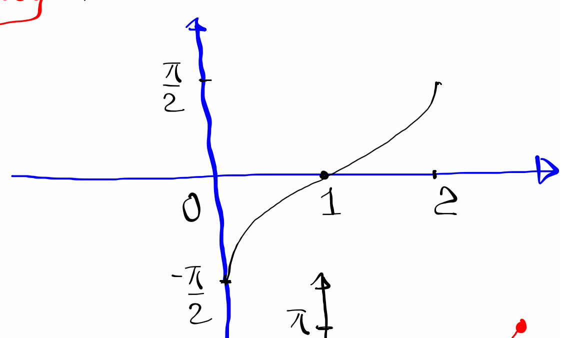
Jan 29-8:25 AM

Graph $y = \sin^{-1}(x-1)$

$-1 \le x-1 \le 1$

Add 1

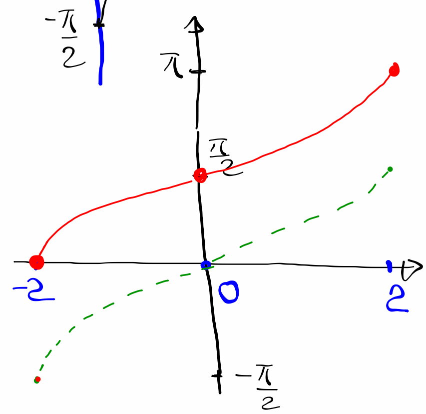
$0 \le x \le 2$



Graph $y = \sin^{-1}\left(\frac{1}{2}x\right) + \frac{\pi}{2}$

$-1 \le \frac{1}{2}x \le 1$

$-2 \le x \le 2$

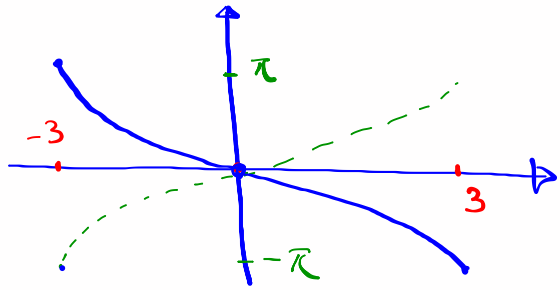


Jan 29-8:33 AM

Graph $y = -2\sin^{-1}\left(\frac{1}{3}x\right)$

$-1 \leq \frac{1}{3}x \leq 1$

$-3 \leq x \leq 3$



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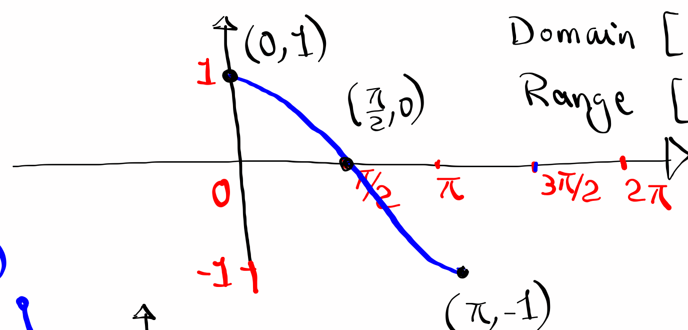
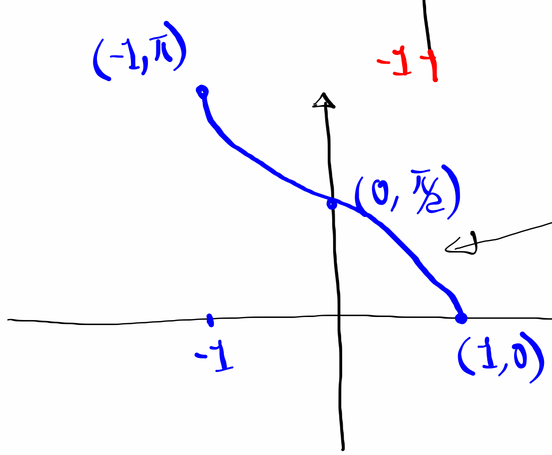
Graph $y = \cos x$

Domain $[0, \pi]$

Range $[-1, 1]$





$\cos^{-1}x$

Domain $[-1, 1]$

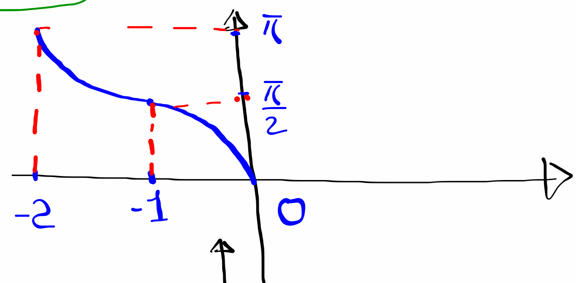
Range $[0, \pi]$

Jan 29-8:44 AM

Graph $y = \cos^{-1}(\underbrace{x+1})$

$-1 \leq x+1 \leq 1$

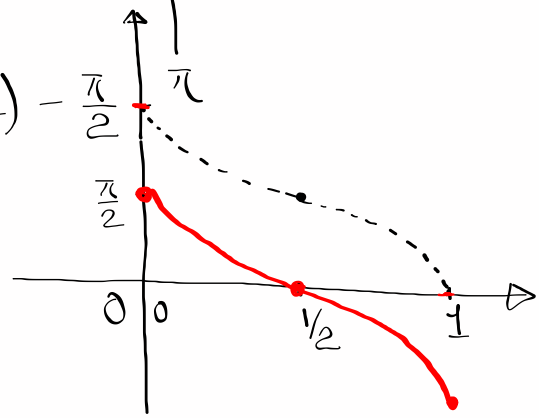
$-2 \leq x \leq 0$



Graph $y = \cos^{-1}(\underbrace{2x-1}) - \frac{\pi}{2}$

$-1 \leq 2x - 1 \leq 1$

$0 \leq x \leq 1$



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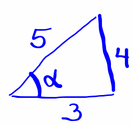
working with expressions with inverse-functions:

Simplify $\sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\alpha = \frac{4}{5}$

Let $\alpha = \cos^{-1}\frac{3}{5}$

$\Rightarrow \cos\alpha = \frac{3}{5}$



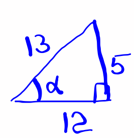
Use your calc in rad. mode.

Sin of $\cos^{-1} .6 = .8$

Simplify $\tan\left(\sin^{-1}\frac{5}{13}\right) = \tan\alpha = \frac{5}{12}$

Let $\alpha = \sin^{-1}\frac{5}{13}$

$\Rightarrow \sin\alpha = \frac{5}{13}$

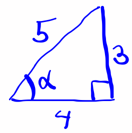


Jan 29-8:58 AM

Simplify $\sin\left(2\cos^{-1}\frac{4}{5}\right) = \sin(2\alpha)$

Let $\alpha = \cos^{-1}\frac{4}{5}$

$\Rightarrow \cos\alpha = \frac{4}{5}$



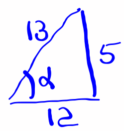
$= 2\sin\alpha\cos\alpha$

$= 2 \cdot \frac{3}{5} \cdot \frac{4}{5}$

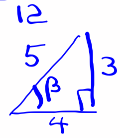
$= \frac{24}{25}$

Simplify $\tan\left(\sin^{-1}\frac{5}{13} - \cos^{-1}\frac{4}{5}\right) = \tan(\alpha - \beta)$

$\alpha = \sin^{-1}\frac{5}{13} \rightarrow \sin\alpha = \frac{5}{13}$



$\beta = \cos^{-1}\frac{4}{5} \rightarrow \cos\beta = \frac{4}{5}$



$= \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \dfrac{\frac{5}{12} - \frac{3}{4}}{1 + \frac{5}{12}\cdot\frac{3}{4}}$

$= \boxed{}$

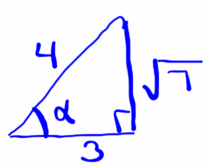
Jan 29-9:05 AM

Find exact value of

$\cos\left(2\cos^{-1}\frac{3}{4}\right)$ Calc. says $.125$

Let $\alpha = \cos^{-1}\frac{3}{4}$

$\cos\alpha = \frac{3}{4}$



$\rightarrow \cos 2\alpha =$

$\cos^2\alpha - \sin^2\alpha =$

$\left(\frac{3}{4}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2 =$

$\frac{9}{16} - \frac{7}{16} = \frac{2}{16}$

$= \frac{1}{8} = \boxed{.125}$

Jan 29-9:13 AM

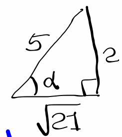
Find $\boxed{\text{exact Value}}$ of

$\sin\left(\frac{1}{2}\sin^{-1}\frac{2}{5}\right)$.

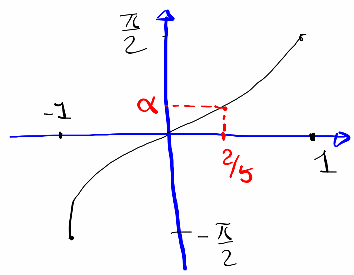
Calc Say .204

Let $\alpha = \sin^{-1}\frac{2}{5}$

$\sin\alpha = \frac{2}{5}$



Look at the graph of $\sin^{-1}x$



$\sin\left(\frac{1}{2}\alpha\right)$

$= \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

$= \pm\sqrt{\dfrac{1-\frac{\sqrt{21}}{5}}{2}}$

$= \pm\sqrt{\dfrac{5-\sqrt{21}}{10}}$

$0 < \alpha < \frac{\pi}{2}$

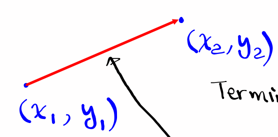
$0 < \alpha/2 < \frac{\pi}{4}$

QI

choose +.

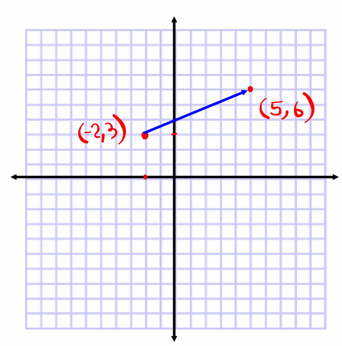
Jan 29-9:19 AM

Vectors: Directed Line Segment



$\vec{u} = \langle x_2 - x_1, y_2 - y_1 \rangle$

Draw the vector $\vec{u}$ with initial Point $(-2,3)$ and terminal point $(5,6)$.



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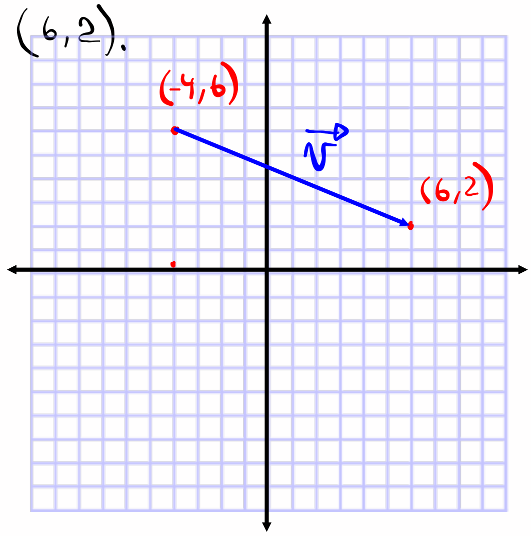
Draw $\vec{v}$ with initial point $(-4,6)$ and terminal point $(6,2)$.



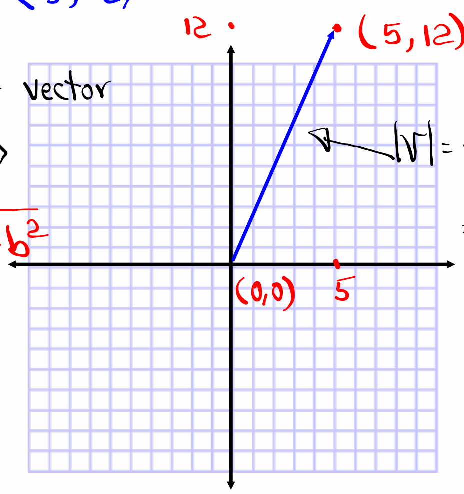
Jan 29-9:49 AM

Vectors in standard position

$\vec{v} = \langle a, b \rangle$ → initial Point $(0,0)$

Terminal point (a,b)

Draw $\vec{v} = \langle 5,12 \rangle$ in standard position.



magnitude of vector

$v = \langle a, b \rangle$

$|v| = \sqrt{a^2+b^2}$

$|v| = \sqrt{5^2+12^2}$

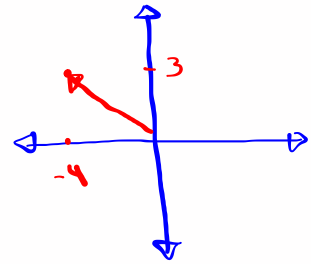
$= \sqrt{169}$

$= \boxed{13}$

Jan 29-9:51 AM

Given $\vec{v} = \langle -4, 3 \rangle$

1) Draw $\vec{v}$ in standard position



2) Find $|v| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = \boxed{5}$

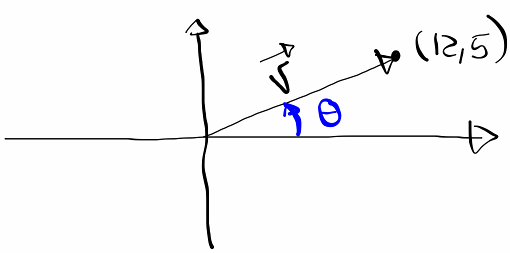
Jan 29-9:56 AM

Direction angle : It is a positive angle between the vector and x-axis.

$\theta \rightarrow$ Direction angle for vector $v = \langle a, b \rangle$

$$\theta = \tan^{-1} \frac{b}{a}$$

Find the direction angle for $v = \langle 12, 5 \rangle$.



$$\theta = \tan^{-1} \frac{b}{a}$$

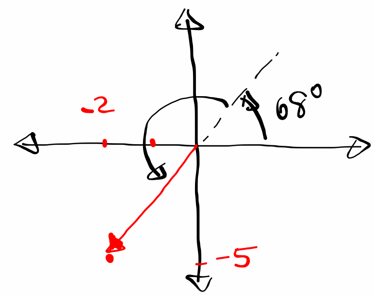
$$= \tan^{-1} \frac{5}{12}$$

$$\approx 23^\circ$$

Jan 29-10:00 AM

Given vector $V = \langle -2, -5 \rangle$

1) Draw in standard position.



2) Find $|V| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

3) Find its direction angle. $\theta = \tan^{-1}\left(\frac{-5}{-2}\right)$

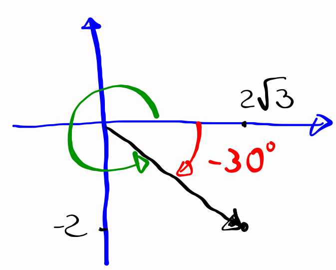
$\approx 68° + 180°$

$\approx \boxed{248°}$

Jan 29-10:05 AM

$V = \langle 2\sqrt{3}, -2 \rangle$

1) Draw in standard position



2) Find $|V| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{4 \cdot 3 + 4} = \sqrt{16} = \boxed{4}$

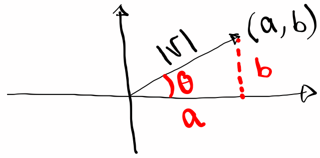
3) Find its direction angle. $\theta \approx \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right)$

$\approx -30° + 360°$

$\approx 330°$

Jan 29-10:10 AM

Expressing vectors using Horizontal and Vertical Components:



$\cos\theta = \frac{a}{|v|}$

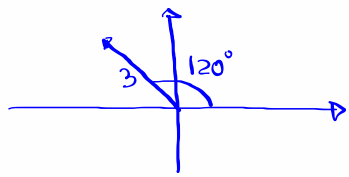
$a = |v|\cos\theta$

$\sin\theta = \frac{b}{|v|}$

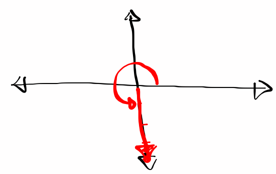
$b = |v|\sin\theta$

$v = \langle a, b\rangle = \langle |v|\cos\theta, |v|\sin\theta\rangle$

$v = \langle 3\cos 120°, 3\sin 120°\rangle$



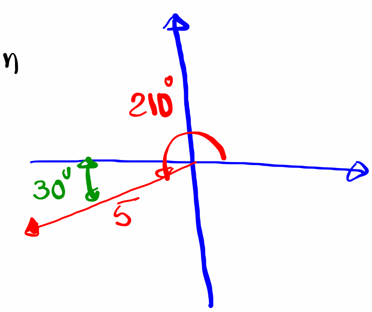
$v = \langle 4\cos 270°, 4\sin 270°\rangle$



Jan 29-10:15 AM

$v = \langle 5\cos 210°, 5\sin 210°\rangle$

1) Draw v in standard position



2) Convert v into $\langle a, b\rangle$

$5\cos 210° = 5 \cdot -\cos 30° = 5 \cdot \frac{-\sqrt{3}}{2} = \frac{-5\sqrt{3}}{2}$

$5\sin 210° = 5 \cdot -\sin 30° = 5 \cdot \frac{-1}{2} = -\frac{5}{2}$

$\langle \frac{-5\sqrt{3}}{2}, -\frac{5}{2}\rangle$

Jan 29-10:22 AM

Operations with Vectors:

$\langle a, b\rangle + \langle c, d\rangle = \langle a+c, b+d\rangle$

$\langle a, b\rangle - \langle c, d\rangle = \langle a-c, b-d\rangle$

$k\langle a, b\rangle = \langle ka, kb\rangle$

$\langle 3,5\rangle + \langle 2,-1\rangle = \langle 5, 4\rangle$

$\langle -1, 4\rangle - \langle 4, 2\rangle = \langle -5, 2\rangle$

$\langle 3, 6\rangle - \langle -5, 6\rangle = \langle 8, 0\rangle$

$5\langle -2, 1\rangle = \langle -10, 5\rangle$

$-3\langle -2, 5\rangle = \langle 6, -15\rangle$

Jan 29-10:27 AM

Dot Product

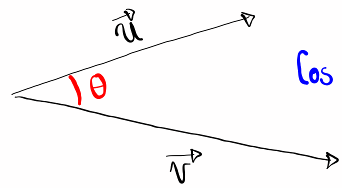
$\langle a, b\rangle \cdot \langle c, d\rangle = ac + bd$

$\langle 4, 3\rangle \cdot \langle 2, 1\rangle = 4\cdot 2 + 3\cdot 1 = \boxed{11}$

$\langle -5, 4\rangle \cdot \langle 4, 5\rangle = -5\cdot 4 + 4\cdot 5 = \boxed{0}$

Jan 29-10:33 AM

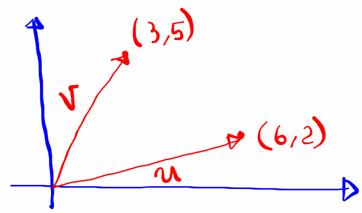
How to find the angle between two vectors:



$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

$u = \langle 6,2 \rangle$, $v = \langle 3,5 \rangle$

1) Draw u & v in standard position.



$u \cdot v = 6 \cdot 3 + 2 \cdot 5 = \boxed{28}$

$|u| = \sqrt{6^2 + 2^2} = \sqrt{40}$

$|v| = \sqrt{3^2 + 5^2} = \sqrt{34}$

$\cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{28}{\sqrt{40}\sqrt{34}}$

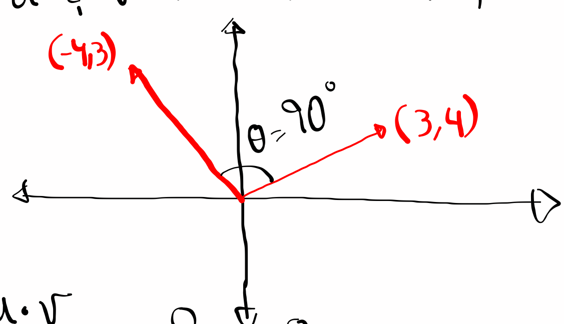
$\cos\theta = .759$

$\theta = \cos^{-1}(.759) \approx \boxed{41^\circ}$

Jan 29-10:37 AM

$u = \langle 3,4 \rangle$, $v = \langle -4,3 \rangle$

1) Draw u & v in standard position.



2) find

$u \cdot v = 0$

$|u| = 5$

$|v| = 5$

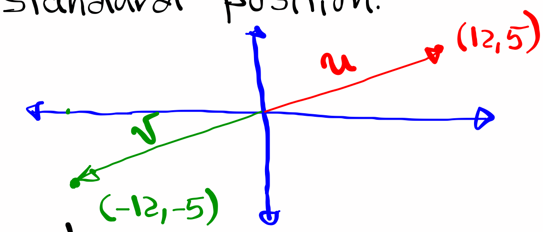
3) $\cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{0}{5 \cdot 5} = \dfrac{0}{25} = 0$

$\theta = \cos^{-1}(0) = \boxed{90^\circ}$

Jan 29-10:45 AM

$u = \langle 12, 5 \rangle \quad v = \langle -12, -5 \rangle$

1) Draw u & v in standard position.



2) Find $|u|$, $|v|$, and $u \cdot v = -169$

$|u| = \sqrt{12^2 + 5^2} = 13$, $|v| = \sqrt{(-12)^2 + (-5)^2} = 13$

3) Find the angle between them.

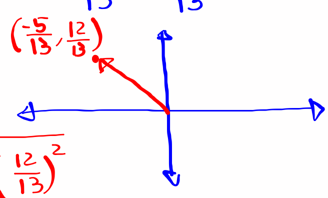
$\cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{-169}{13 \cdot 13} = -1$ $\quad \theta = \cos^{-1}(-1)$

$\boxed{\theta = 180°}$

Jan 29-10:51 AM

unit Vector → Magnitude = 1

Verify that $u = \langle \frac{-5}{13}, \frac{12}{13} \rangle$ is a unit vector.



$|u| = \sqrt{\left(\frac{-5}{13}\right)^2 + \left(\frac{12}{13}\right)^2}$

$= \sqrt{\frac{25}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = \sqrt{1} = \boxed{1}$

Special unit vector:

$i = \langle 1, 0 \rangle$, $j = \langle 0, 1 \rangle$

$\langle 3, -2 \rangle = \langle 3, 0 \rangle + \langle 0, -2 \rangle$

$= 3\langle 1, 0 \rangle - 2\langle 0, 1 \rangle$

$= \boxed{3i - 2j}$

Jan 29-11:22 AM

$u = 4i + 3j$

$= 4\langle 1, 0\rangle + 3\langle 0, 1\rangle$

$= \langle 4, 0\rangle + \langle 0, 3\rangle$

$= \langle 4, 3\rangle$

$v = ai + bj$

$= \langle a, b\rangle$

$u = 5i - 2j$

$v = 2i + 4j$

$\boxed{u + v = 7i + 2j}$

$-2u = -2(5i - 2j)$

$= \boxed{-10i + 4j}$

$|v| = \sqrt{2^2 + 4^2} = \sqrt{20}$

$u \cdot v = \langle 5, -2\rangle \cdot \langle 2, 4\rangle = 5 \cdot 2 + -2 \cdot 4 = \boxed{2}$

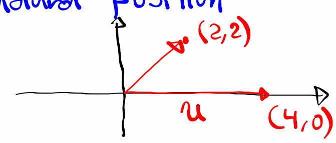
Jan 29-11:27 AM

Given $u = 4i$, $v = 2i + 2j$

1) Draw u & v in standard position

$4i = \langle 4, 0\rangle$

$2i + 2j = \langle 2, 2\rangle$



2) Find $|u|$, $|v|$, and $u \cdot v$

$|u| = \sqrt{4^2 + 0^2} = 4$, $|v| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$u \cdot v = 4 \cdot 2 + 0 \cdot 2$

$= \boxed{8}$

3) Find the angle between u & v.

$\cos\theta = \dfrac{u \cdot v}{|u||v|} \Rightarrow \cos\theta = \dfrac{\cancel{8}^{\,2}}{\cancel{4} \cdot \sqrt{8}}$

$= \dfrac{\cancel{2}}{\sqrt{\cancel{4}}\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2} \rightarrow \boxed{\theta = 45^\circ}$

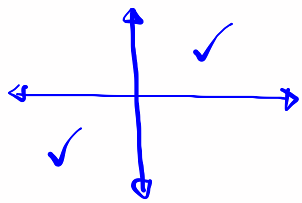
Jan 29-11:33 AM

Solve $\tan x - \sqrt{3} = 0$ on $[0, 2\pi)$.

$\tan x = \sqrt{3}$

QI Angle = RA + $n \cdot$ Period

QIII Angle = π + R.A. + $n \cdot$ Period



QI $\boxed{x = \frac{\pi}{3} + n \cdot \pi}$

R.A. $60°$, $\frac{\pi}{3}$ QIII $x = \pi + \frac{\pi}{3} + n \cdot \pi$

$= \boxed{\frac{4\pi}{3} + n\pi}$

$n = 0 \rightarrow \pi/3, \frac{4\pi}{3}$

$n = 1 \rightarrow \cancel{\frac{4\pi}{3}}, \cancel{\frac{7\pi}{3}}$ (Passed 2π)

$\left\{ \frac{\pi}{3}, \frac{4\pi}{3} \right\}$

Jan 29-11:40 AM

Solve $\tan^2 2x - 1 = 0$ on $[0°, 360°)$

$\tan^2 2x = 1$

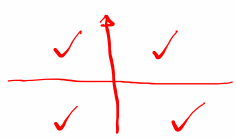
$\tan 2x = \pm 1$

QI Angle = RA + $n \cdot 180°$

QII Angle = $180°$ − RA + $n \cdot 180°$

QIII Angle = $180°$ + RA + $n \cdot 180°$

QIV Angle = $360°$ − RA + $n \cdot 180°$



R.A. $45°$

$2x = 45° + n \cdot 180° \longrightarrow x = 22.5° + n \cdot 90°$

$2x = 180° - 45° + n \cdot 180° \longrightarrow x = 67.5° + n \cdot 90°$

$2x = 180° + 45° + n \cdot 180° \longrightarrow x = 112.5° + n \cdot 90°$

$2x = 360° - 45° + n \cdot 180° \longrightarrow x = 157.5° + n \cdot 90°$

$n = 0 \rightarrow 22.5°, 67.5°, 112.5°, 157.5°$

$n = 1 \rightarrow \cancel{112.5°}, \cancel{157.5°}, 202.5°, 247.5°$

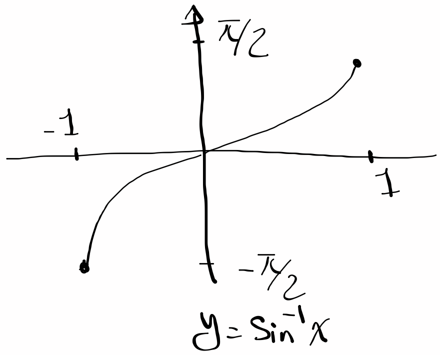
$n = 2 \rightarrow \cancel{202.5°}, \cancel{247.5°}, 292.5°, 337.5°$

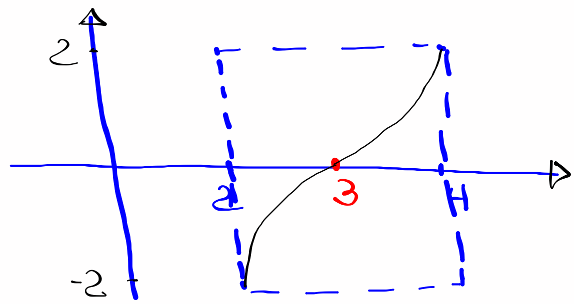
Jan 29-11:46 AM

Graph $y = \frac{4}{\pi} \sin^{-1}(x-3)$

$-1 \le x-3 \le 1$

$2 \le x \le 4$





$\frac{4}{\pi} \cdot \frac{-\pi}{2} = -2$

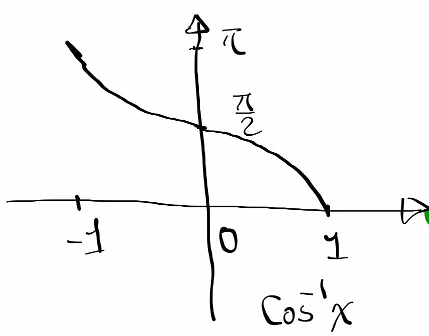
$\frac{4}{\pi} \cdot \frac{\pi}{2} = 2$

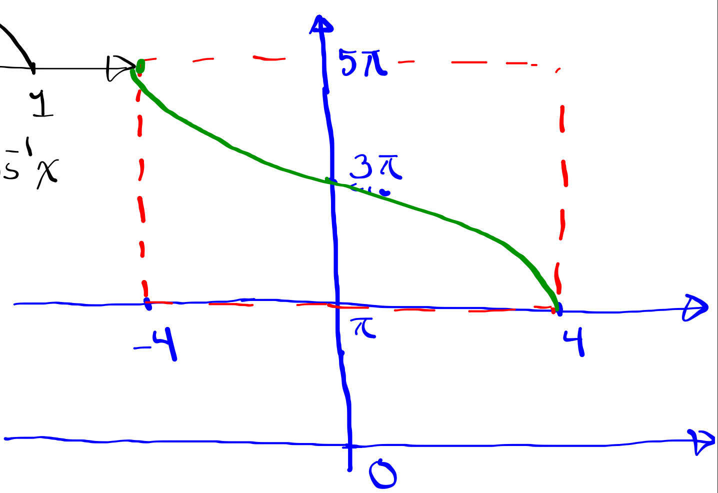
Jan 29-11:56 AM

Graph $y = 4 \cos^{-1}\left(\frac{1}{4}x\right) + \pi$

$-1 \le \frac{1}{4}x \le 1$

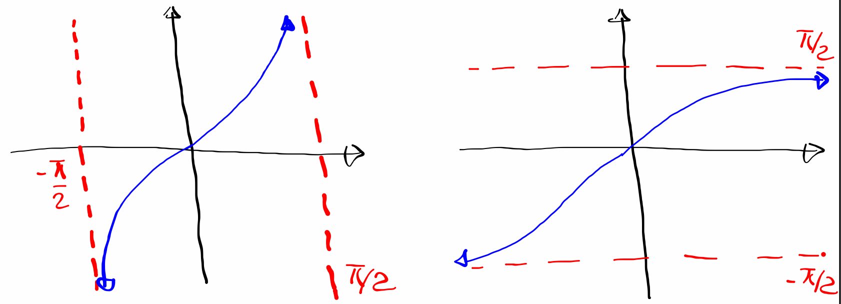
$-4 \le x \le 4$



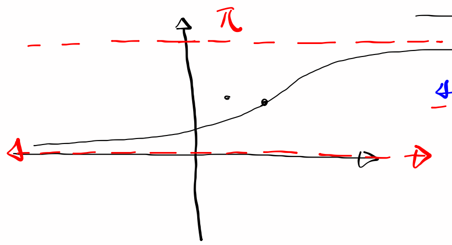


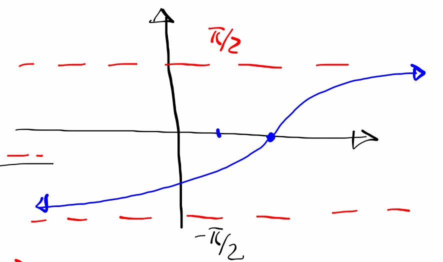
Jan 29-12:00 PM

$y = \tan x$ $\qquad$ $y = \tan^{-1} x$



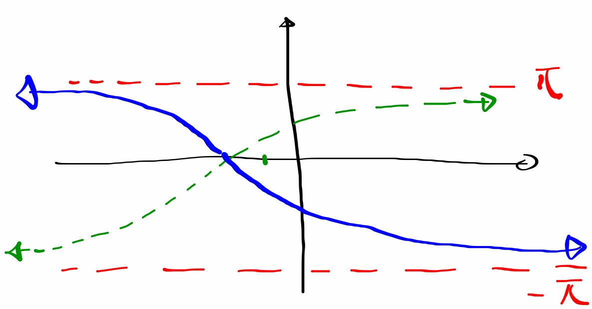
$y = \tan^{-1}(x-2) + \frac{\pi}{2}$





Jan 29-12:05 PM

Graph $\quad y = -2\tan^{-1}(x+2)$



Jan 29-12:11 PM

Simplify $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right) = \sin(30° + 60°)$

$\alpha = \sin^{-1}\frac{1}{2}$

$\sin\alpha = \frac{1}{2}$

$\alpha = 30°$

$\beta = \cos^{-1}\frac{1}{2}$

$\cos\beta = \frac{1}{2}$

$\beta = 60°$

$= \sin 90°$

$= \boxed{1}$

Simplify $\sin\left(2\tan^{-1} 1\right) = 1$

$\alpha = \tan^{-1} 1$

$\tan\alpha = 1$

$\alpha = 45°$

$\sin(2 \cdot 45°) = \sin 90°$

$= \boxed{1}$

Jan 29-12:15 PM